Resource theories

a generalized framework

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Resource theories

The big question

What can we achieve with some resources and some operations?

...in quantum information theory

- Local operations and classical communication
- Thermodynamics: Gibbs-preserving maps, thermal operations, noisy operations

...in life

- chemistry
- markets
- games (e.g. Lego, Minecraft, Agricola)

Resource theories

Common elements

- Set of allowed operations
- Free states (thermal states, separable states)
- Pre-order structure on state space
- Monotones (free energy, entropy, entanglement measures)
- Scalable currency (pure qubits, Bell pairs, wits)
- An agent

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In this talk

- Thinking about knowledge
- General framework for resource theories
- Locality aspects of resource theories

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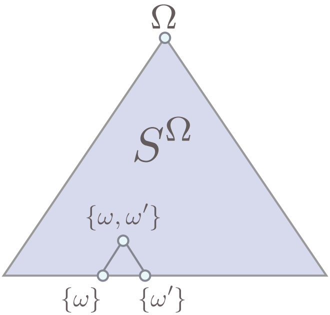
- Elephant = {Asian elephant, African elephant}.
- ρ_A = all the states that have marginal ρ in subsystem A.

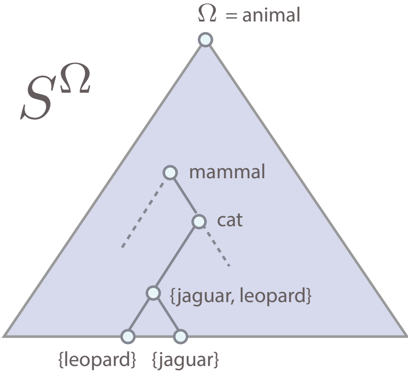
Specification space



Formalism

- State space Ω: elements $\omega \in \Omega$ correspond to most precise descriptions of reality.
- Specifications $V \subseteq \Omega$ correspond to states of knowledge. E.g. $V = \{\omega, \omega'\}$.
- Specification space S^Ω: all subsets V of Ω.
- Ordered by inclusion $\subseteq$: more specific description.

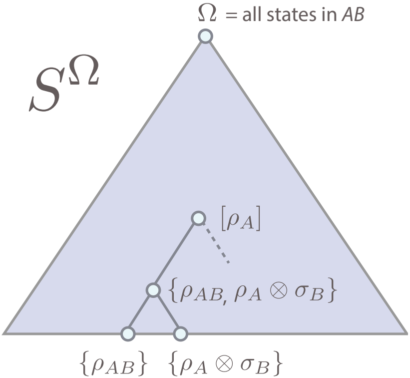
Specification space



Example: knowledge about animals

- State space Ω: all animal species
- Specifications $V \subseteq \Omega$, e.g. $V = \{$leopard, jaguar, cheetah, Iberian lynx, ...$\}$ = cat.

Specification space



Example: knowledge about quantum states

- State space Ω: all quantum states (density matrices) on bipartite system $A \otimes B$.
- Specifications $V \subseteq \Omega$, e.g. $V = \{\rho_{AB}, \sigma_{AB}\}$ or $[\rho_A] = \{\omega \in \Omega : \mathrm{Tr}_B\, \omega = \rho_A\}$.

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- $[\rho_A \otimes \sigma_B] = [\rho_A] \cap [\sigma_B] \cap \bigotimes_{AB}$

Resource theories

$$(\mathcal{S}^{\Omega}, \mathcal{T})$$

Allowed operations

- $\mathcal{T}$ monoid of allowed operations.
- Gibbs-preserving maps, noisy operations, LOCC, etc.
- Rules of a game, trading an item for one of lesser value.

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Constructions

- Constructions: $V \xrightarrow{f} W$, if $f(V) \subseteq W$.
- $f(\text{parrot}) = \text{cat} \subseteq \text{mammal}$.
- Induces a pre-order structure on the specification space:
  $V \to W : \quad \exists\, f \in \mathcal{T} : V \xrightarrow{f} W$.
- Monotones: if $V \to W$, then $M(V) \geq M(W)$.

Resource theories

Free and conserved resources

- V is free if $\Omega \to V$.
- Global Gibbs state (thermal operations, Gibbs-preserving maps), product states (local operations).
- V is conserved if all allowed operations satisfy $f(V) = V$.
- Global Gibbs state (thermal ops.), $\bigotimes_{AB}$ (local ops.).

Convex resource theories

- Theory convex if Ω convex and all $f \in \mathcal{T}$ preserve convexity.
- Theory doubly convex if $\mathcal{T}$ also convex.
- Theory doubly convex $\implies$ set of free resources convex.

Resource theories

Combining resource theories

- $(\mathcal{S}^\Omega, \mathcal{T}), (\mathcal{S}^\Omega, \mathcal{F}) \to (\mathcal{S}^\Omega, \mathcal{T} \cap \mathcal{F})$.
- Thermal operations + LOCC.
- $V \to W$ in $(\mathcal{S}^\Omega, \mathcal{T})$ if it is possible in both theories.

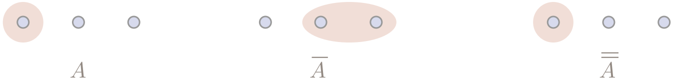
Relating resource theories

- $(\mathcal{S}^\Omega, \mathcal{T})$ more restrictive than $(\mathcal{S}^\Omega, \mathcal{M})$ if $\mathcal{T} \subset \mathcal{M}$.
- $\mathcal{M}$ all TPCPMs, $\mathcal{T}$ thermal operations.
- Daughter obtained via conservation laws if

$$\exists\ V \subseteq \Omega : \quad \mathcal{T} = \{f \in \mathcal{M} : f(V) = V,\ \forall\ V \in \mathcal{V}\}.$$

Subsystems

- Defined operationally from commutation relations of transformations in the mother theory $(S^{\Omega}, \mathcal{M})$.



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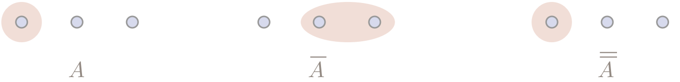
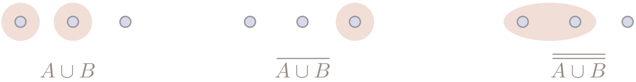


- $A \subseteq \mathcal{M}$ subsystem if $\overline{\overline{A}} = A$.

$$\overline{A} = \{g \in \mathcal{M} : gf(V) = fg(V),\ \forall\ f \in A, V \in S^{\Omega}\}.$$



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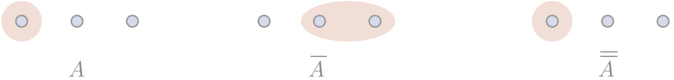

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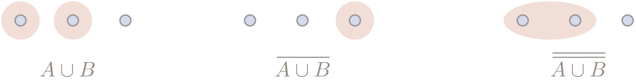

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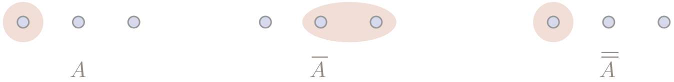

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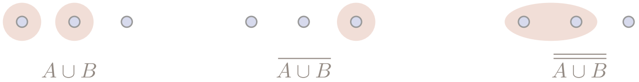

- Joint systems: $A \vee B := \overline{\overline{A \cup B}}$.
- Daughter theory $(S^{\Omega}, \mathcal{T})$ inherits subsystem structure.

Local specifications

- Generalized partial trace:

$$\text{For}_A: \quad S^\Omega \mapsto S^\Omega$$
$$V \to \text{For}_A(V) \supseteq V: \quad \text{For}_A(g_{\overline{A}} V) = g_{\overline{A}}(\text{For}_A(V)).$$

$$\text{Tr}_A(V) = \bigcup_i \text{For}^i_A(V)$$

- If the mother is quantum theory (TPCPMs), we recover the partial trace.
- Allows us to define local resource theories.
- Subsystem independence $\bigotimes_{AB}$ preserved by local operations:

$$\bigcup_{f_A \in A} \bigcup_{g_B \in B} f_A \circ \ g_B \left(\bigotimes_{AB} \right) = \bigotimes_{AB}$$

Currency

Copies of local specifiations

- Start with V_A local in A
- Swap subsystems A and A' to obtain $V_{A'}$
- Combine $V_A \cap V_{A'} \cap \bigotimes_{AA'} = {V_A}^{\otimes 2}$

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Currency

- C currency if given enough copies, we can always perform a transformation $V \otimes C^{\otimes n} \to W$.
- $\min n$: cost of transformation $V \to W$ in terms of currency C.
- yield of transformation $V \to W$: $\max n : V \to W \otimes C^{\otimes n}$.
- Can be used to define monotones in modular resource theories.

Related work

- Thermo: Brandão, Horodecki, Oppenheim, Renes and Spekkens (2011), Gour, Müller, Narasimhachar, Spekkens, Yunger Halpern (2014).
- Single-shot thermo: Aberg (2013), Oppenheim and Horodecki (2013), Faist, Dupuis, Oppenheim, Renner (2012), Brandão, Horodecki, Ng, Oppenheim, Wehner (2013).
- Pre-order: Lieb and Yngvason (1998, 1999, 2013), Weilenmann, Krämer, Faist, Renner (2015).
- Cryptography: Maurer and Renner (2011).
- Categories: Coecke, Fritz, Spekkens (2014).
- Quantum: Brandão, Gour (2015).

Conclusions

This talk

- Specification space: systematic treatment of knowledge.
- Resource theories: defining, combining and deriving them.
- Operational subsystems: defined from modularity of transformations.
- Currency: definition and monotones.

Also studied

- Embeddings (map different versions of reality).
- Approximation structures (notion of distances in Ω).
- Approximate transformations.

Directions

- Exploring monotones.
- Applications beyond quantum theories.

Convexity

What about probability?

- Ω convex: $\omega, \omega' \in \Omega \implies p\, \omega + (1-p)\, \omega' \in \Omega$.
- $V^{\mathbb{P}}$ convex hull of V: smallest convex set that contains V

$$V^{\mathbb{P}} = \bigcup_{P_V \in \mathbb{P}_V} \left\{ \sum_{\omega \in V} P_V(\omega)\, \omega \right\}$$

- $\{\rho, \sigma\}^{\mathbb{P}} = \bigcup_p \{p\, \rho + (1-p)\, \sigma\}$.
- Probabilistic equivalence: $V \sim V^{\mathbb{P}}$.

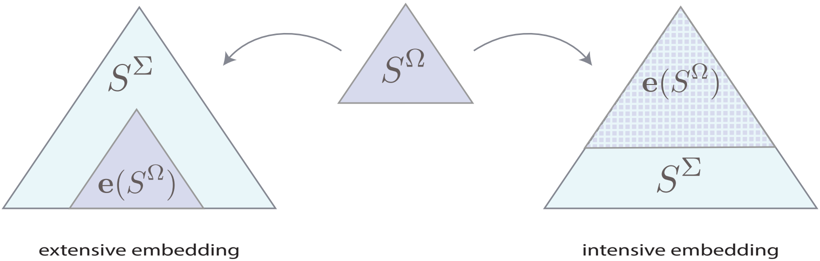
Transformations

Homomorphisms

$$f : S^{\Omega} \mapsto S^{\Sigma}$$
$$V \to \bigcup_{\omega \in V} f(\{\omega\}).$$

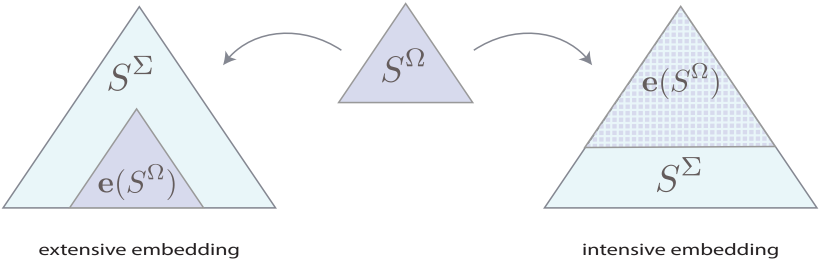
- Quantum operations: $f_{\mathcal{E}}(\{\rho, \sigma\}) = \{\mathcal{E}(\rho)\} \cup \{\mathcal{E}(\sigma)\}$.

Embeddings



- Another observer may have a more comprehensive description of reality. Embeddings conciliate the two views.
- Extensive embeddings: more elements of reality, $\Omega \subset \Sigma$.
- Intensive embeddings: more precise descriptions of $\omega \in \Omega$.

Embeddings



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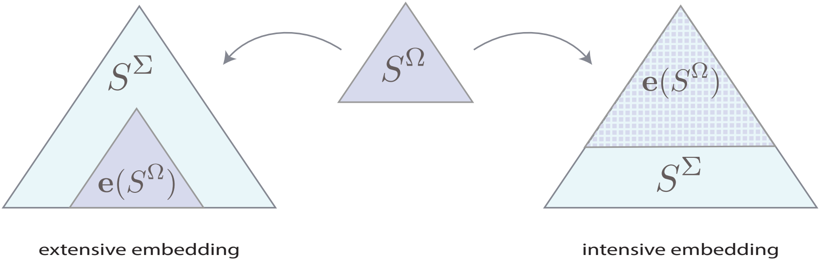
- Extensive embeddings: add new species $\omega =$ *Saltuarius eximius*.
- Intensive embeddings: able to distinguish gender: $\mathbf{e}$(leopard) $= \{$male leopard, female leopard$\}$.

Embeddings



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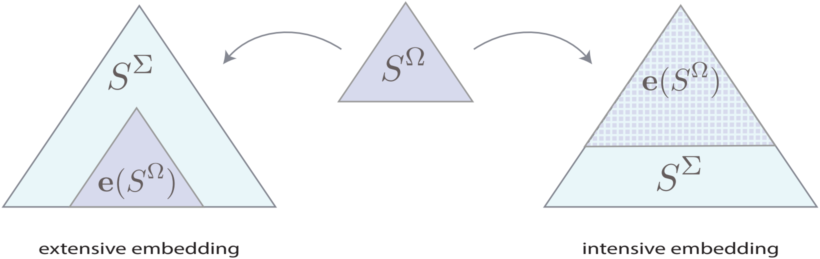

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- Extensive embeddings: from real density matrices to complex ones.
- Intensive embeddings: hidden variables, larger Hilbert space.